**B. Math. (Hons.) First year**
**First Midsemestral examination**
**10th September 2018**
**Analysis I**
**Instructor : B. Sury**

**Each question carries 12 marks.**

**Q 1.**
(a) Determine the infimum and supremum of the set $\{\sin(n\pi/3) : n \in \mathbf{Z}\}$ and find a sequence from this set that converge to the infimum.
(b) If $S, T$ are bounded subsets of real numbers, prove that the supremum of the set $\{s + t : s \in S, t \in T\}$ equals $sup(S) + sup(T)$.

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**Q 2.**
(a) Let $a, b > 0$. Find the limit of the sequence $(a^n + b^n)^{1/n}$ as $n \to \infty$.
(b) Prove that the sequence $\{\sin(n)\}_n$ has an (infinite) subsequence $\{\sin(n_k)\}_k$ which is completely contained in $]1/2, 1]$.

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**Q 3.** Consider the sequence defined recursively by $x_1 = 1, x_{n+1} = \frac{x_n^3+1}{4}$ for all $n \geq 1$.
(i) Prove that $\{x_n\}$ is a decreasing, positive sequence bounded below.
(ii) Deduce that there exists $c \in (0, 1)$ such that $|x_{n+1} - x_n| < c/2$ for $n > 1$.
(iii) Deduce that $\{x_n\}$ is a Cauchy sequence.
(iv) State what the above process yields in terms of roots of the polynomial $x^3 - 4x + 1$.

**OR**

For a positive integer $n$, consider the arithmetic and geometric means of the $n + 1$ numbers $1 + 1/n$ (repeated $n$ times) and 1 to deduce that the sequence $a_n = (1+1/n)^n$ is monotonically increasing. Similarly, for $n > 1$, looking at the arithmetic and geometric means of the $n+1$ numbers $1-1/n$ (repeated $n$ times) and 1, deduce that $b_n = (1-1/n)^{-n}$ is monotonically decreasing. Finally, find a relation between $b_{n+1}$ and $a_n$ to deduce that both sequences $\{a_n\}$ $\{b_n\}$ converge and, converge to the same limit.

**OR**

If $\{a_n\}$ is a sequence of positive, real numbers such that the sequence $a_{n+1}/a_n$ converges, prove that the sequence $a_n^{1/n}$ also converges, and converges to the same limit. Give an example to show that the converse may not be true.

**Q 4.** Prove that the series $\sum_{n=2}^{\infty} \frac{1}{n\log(n)}$ diverges and that $\sum_{n=2}^{\infty} \frac{1}{n\log(n)^{1.1}}$ converges.
*Hint:* You may prove and use the following result of Cauchy:
If $a_n > 0$ and $\{a_n\}$ is decreasing, then

$$\frac{1}{2}\sum_{k=0}^{n} 2^k a_{2^k} \leq \sum_{r=1}^{n} a_r \leq \sum_{k=0}^{n-1} 2^k a_{2^k} + a_{2^n}.$$

**OR**

Consider the series $\sum_{n\geq 1} a_n$ where $a_n = 2^{(-1)^n n}$.
Determine $\lim inf|a_n|^{1/n}, \lim sup|a_n|^{1/n}, \lim inf|a_{n+1}/a_n|, \lim sup|a_{n+1}/a_n|$. What does root test give? What does ratio test give?

**OR**

Using root/ratio/Raabe tests or otherwise, determine the convergence or otherwise of each of the following series (don't find the sum!):
(a) $\sum_{n\geq 1} \frac{1}{\binom{2n}{n}}$;
(b) $\sum_{n\geq 1} a_n^3$ where $a_n = \frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots(2n)}$.

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**Q 5.**
(a) Define the interior $S^0$ and closure $\bar{S}$ of a set $S \subset \mathbf{R}$.
(b) If $S_n (n \geq 1)$ are subsets of $\mathbf{R}$, then prove that $(\int_n S_n)^0 \subseteq \int_n S_n^0$. Give an example to show that the inclusion could be proper.
(c) For any subset $S$ of $\mathbf{R}$, prove that $S^0 = (\bar{S^c})^c$ where $A^c$ denotes the complement of $A$.

**OR**

(a) Write down a set of open intervals $I_n (n \geq 1)$ which cover $(0,1)$ such that finitely many of the $I_n$'s do not cover $(0,1)$.
(b) Suppose $S$ is a set of real numbers such that whenever $S \subset \bigcup_{n\geq 1} U_n$ with $U_n$'s open, there is a positive integer $n$ such that $S \subset \bigcup_{m=1}^{n} U_m$. Prove that $S$ must be closed.
*Hint:* Show that $S^c$ is open.